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DYNAMIC STABILITY OF COLUMNS SUBJECTED TO NONCONSERVATIVE FORCES

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October 1979

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The numerical results of a class of problems of linear elastic stability problems subjected to nonconservative forces and under various support conditions are presented here. A single solution formulation by which these results have been obtained is described. Accuracy of these results compared with those reported in the literature is discussed.

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I. INTRODUCTION. Any particular subject of investigation in applied sciences is always motivated by the desire to understand some natural phenomena and hopefully to utilize the results of such an investigation for the benefit of human activities. The study of structural behavior under nonconservative loads is of no exception. Since follower forces are a special class of nonconservative forces [1], one is surprised to encounter frequently the question as to the relation between such a study and a real engineering problem. Physically, a follower force is simply one whose direction follows the structural deformation as in comparison with a dead load which acts in a fixed direction independent of deformation. Some obvious examples of follower forces are: thrust at the tail of a flexible rocket, jet engine thrust of an airplane, thrust on the propeller shaft of a ship, etc. Other examples such as the pressure-and-curvature induced forces included in the gun dynamic studies are less obvious [2].

Since the problems of follower forces are non-self-adjoint their treatment is more difficult than that for the self-adjoint problems. In the classical paper by Beck [3], it was demonstrated that the stability nature of a nonconservative problem can be quite different than that of a conservative one. For these reasons, a systematic approach to this class of problems and an understanding of some of the basic problems involving follower forces are desirable.

The purpose of this paper is to present a single solution approach to a class of problems of follower forces, including several classical examples, to present the numerical results so obtained, and to discuss the accuracy compared with those already published in literature.

In Section II, the class of problems will be defined by a general form of a differential equation and a set of boundary conditions. The solution formulation and its basis are given in Section III. Numerical results of some specific problems are given in Section IV together with a discussion and comparisons with data available in literature.

II. A CLASS OF PROBLEMS SUBJECTED TO FOLLOWER LOADS. The class of problems considered in this paper can be described by the differential equation

$$y'''' + P(x)y'' + \lambda^2 y = 0 \tag{1}$$

where $y(x)$ denotes the lateral disturbance of a beam, as a function of the abscissa x, $P(x)$ is the axial force always tangent to the deformed axis, and λ is the eigenvalue. As usual, a prime denotes differentiation with respect to x.

Eq. (1) is a non-self-adjoint differential equation (thus nonconservative problem) except for $P(x)$ = constant. If the axial force $P(x)$ remains fixed in the direction of the undeformed axis, the problem would be of conservative nature and the differential equation a self-adjoint one.

$$y'''' + [P(x)y']' + \lambda^2 y = 0 \tag{1'}$$

Both Eqs. (1) and (1') are well known and the derivations are simple and they follow the procedures given in such textbooks as that by Timoshenko and Gere [4]. Boundary conditions considered will be in the following form:

$$y'''(0) + P(0)y'(0) + k_1(0) = 0 \tag{2a}$$

$$-y''(0) + k_2 y'(0) = 0 \tag{2b}$$

$$-y'''(1) - (1-k_5)P(1)y'(1) + k_3 y(1) = 0 \tag{2c}$$

$$y''(1) + k_4 y'(1) = 0 \tag{2d}$$

where k_1, k_2 are the deflection and rotation spring constants at $x = 0$ and k_3, k_4 are the same at $x = 1$. The constant k_5 is related to a "constant of tangency" K_θ by equation

$$K_\theta = k_5 - 1 \tag{3}$$

so that Eq. (2c) becomes

$$-y'''(1) + K_\theta P(1)y'(1) + k_3 y(1) = 0 \tag{2c'}$$

where now, if $P(1) \neq 0$, $\theta = K_\theta y'(1)$ denotes the angle that $P(1)$ is to be rotated with respect to the tangent of the beam at $x = 1$ (Figure 1).

Eqs. (2) simply state that the total shear force and moment at $x = 0$ and $x = 1$ must be zero. As k_1 approaches infinity, Eq. (2a) requires that $y(0) = 0$. Thus a zero deflection boundary condition is arrived at. Similar options are provided for by other spring constants k_2, k_3 and k_4.

Three different $P(x)$ will be considered in this paper: (1) $P(x) = P$, a constant, (2) $P(x) = q(1-x)$, and (3) $P(x) = q_0/2(1-x)^2$ where P represents a concentrated force at $x = 0$, q is a uniformly distributed follower force density and q_0 denotes the maximum of a linearly varied follower force density. With the special boundary conditions of a cantilever, case (1), (2), and (3) become the classical problems first solved by Beck [3], Leipholz [5], and Hanger [6], respectively.

III. <u>SOLUTION FORMULATIONS</u>. The solution method used here is the finite element unconstrained variational formulation which has proved to be efficient and simple to use for solutions of non-self-adjoint problems [7,8]. Finite elements are used in the usual sense that the unknown function is approximated by piecewise cubic splines. An unconstrained variational statement is established and used so that none of the boundary conditions need to be satisfied a priori. An outline of the formulation will be given here.

Introducing an adjoint field variable $y^*(x)$, it is a simple matter to see that the following variational statement will lead to the differential equation (1) and boundary conditions (2):

$$\delta I(y,y^*) = 0 \tag{4a}$$

$$I = \int_0^1 (y''y^{*''} - P(x)y'y^{*'} - P'(x)y'y^* + \lambda^2 yy^*)dx$$
$$+ k_1 y(0)y^*(0) + k_2 y'(0)y^{*'}(0) + k_3 y(1)y^*(1) + k_4 y'(1)y^{*'}(1)$$
$$+ k_5 P(1)y'(1)y^*(1) \ . \tag{4b}$$

The fact that Eqs. (4) lead to the given differential equation and boundary conditions for $y(x)$ independent of $y^*(x)$ implies that one can take the variation of I at $y^*(x) \equiv 0$ and $(\delta I)_{y^*\equiv 0} = 0$ still leads to the original problem. Hence our formulation begins with

$$(\delta I)_{y^*\equiv 0} = 0 \tag{5a}$$

or,
$$\int_0^1 [y''\delta y^{*''} - P(x)y'\delta y^{*'} - P(x)y'\delta y^* + \lambda^2 y\delta y^*]dx$$
$$+ k_1 y(0)\delta y^*(0) + k_2 y'(0)\delta y^{*'}(0) + k_3 y(1)\delta y^*(1) + k_4 y'(1)\delta y^{*'}(1)$$
$$+ k_5 y'(1)\delta y^*(1) = 0 \tag{5b}$$

Finite element discretization enters when the beam is divided into L equal elements and Eq. (5b) is written as

$$\sum_{i=1}^{L} \int_0^1 [L^3 y^{(i)''}\delta y^{*(i)''} - LP^{(i)}(\xi)y^{(i)'}\delta y^{*(i)'}$$
$$- LP^{(i)'}y^{(i)'}\delta y^{*(i)} + \frac{\lambda^2}{L} y^{(i)}\delta y^{*(i)}]d\xi$$
$$+ k_1 y^{(1)}(0)\delta y^{*(1)}(0) + k_2 L^2 y^{(1)'}(0)\delta y^{*(1)'}(0)$$
$$+ k_3 y^{(L)}(1)\delta y^{*(L)}(1) + k_4 L^2 y^{(L)'}(1)\delta y^{*(L)'}(1)$$
$$+ k_5 y^{(L)'}(1)\delta y^{*(L)}(1) = 0 \tag{6}$$

In obtaining Eq. (6) from (5b), one has effected a change of coordinates from x (global) to ξ (local) such that

$$\begin{aligned} \xi = \xi^{(i)} &= Lx-i+1 \\ d\xi &= Ldx \\ y(x) &= y^{(i)}(\xi) \\ y'(x) = \frac{d}{dx}y(x) &= L\frac{d}{d\xi}y^{(i)}(\xi) = Ly^{(i)'}(\xi) \end{aligned} \tag{7}$$

etc.

Introducing generalized coordinates vector $\underset{\sim}{Y}^{(i)}$ and shape function vector $\underset{\sim}{a}(\xi)$ such that

$$y^{(i)}(\xi) = \underset{\sim}{a}^T(\xi)\underset{\sim}{Y}^{(i)} \tag{8a}$$

with
$$\underset{\sim}{Y}^{(i)T} = \{Y_1^{(i)} \quad Y_2^{(i)} \quad Y_3^{(i)} \quad Y_4^{(i)}\} \tag{8b}$$

$$\underset{\sim}{a}(\xi) = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{Bmatrix} \tag{8c}$$

where a superscript T denotes the transpose of a matrix. One observes that

$$Y_1^{(i)} = y^{(i)}(0) \quad , \quad Y_2^{(i)} = y^{(i)\prime}(0)$$
$$Y_3^{(i)} = y^{(i)}(1) \quad , \quad Y_4^{(i)} = y^{(i)\prime}(1) \tag{8d}$$

The counterparts for $y^*(\xi)$ can be similarly defined.

In terms of $\underset{\sim}{Y}^{(i)}$, $\underset{\sim}{Y}^{*(i)}$, $\underset{\sim}{a}$, Eq. (6) can be written as:

$$\sum_{i=0}^{L} \delta Y^{*(i)T}\{L^3\int_0^1 \underset{\sim}{a}''(\xi)\underset{\sim}{a}''^T(\xi)d\xi - L\int_0^1 P^{(i)}(\xi)\underset{\sim}{a}'(\xi)\underset{\sim}{a}'^T(\xi)d\xi$$
$$- L\int_0^1 P^{(i)\prime}(\xi)\underset{\sim}{a}(\xi)\underset{\sim}{a}'^T(\xi)d\xi + \frac{\lambda^2}{L}\int_0^1 \underset{\sim}{a}(\xi)\underset{\sim}{a}^T(\xi)d\xi\}\underset{\sim}{Y}^{(i)}$$
$$+ \delta\underset{\sim}{Y}^{*(1)T}\{k_1\underset{\sim}{a}(0)\underset{\sim}{a}^T(0) + k_2L^2\underset{\sim}{a}'(0)\underset{\sim}{a}'^T(0)\}\underset{\sim}{Y}^{(1)}$$
$$+ \delta\underset{\sim}{Y}^{*(L)T}\{k_3\underset{\sim}{a}(1)\underset{\sim}{a}^T(1) + k_4L^2\underset{\sim}{a}'(1)\underset{\sim}{a}'^T(1) + k_5\underset{\sim}{a}(1)\underset{\sim}{a}'^T(1)\}\underset{\sim}{Y}^{(L)} = 0 \tag{9}$$

It will be convenient to define the following matrices:

$$\underset{\sim}{A}_1 = \int_0^1 a(\xi)a^T(\xi)d\xi \quad , \quad \underset{\sim}{A}_2 = \int_0^1 a'(\xi)a'^T(\xi)d\xi$$
$$\underset{\sim}{A}_3 = \int_0^1 a''(\xi)a''^T(\xi)d\xi \quad , \quad \underset{\sim}{A}_4 = \int_0^1 a(\xi)a'^T(\xi)d\xi$$
$$\underset{\sim}{A}_5 = \int_0^1 \xi a'(\xi)a'^T(\xi)d\xi \quad , \quad \underset{\sim}{A}_6 = \int_0^1 a(\xi)a'^T(\xi)d\xi$$
$$A_7 = \int_0^1 \xi^2 a'(\xi)a'^T(\xi)d\xi \tag{10}$$

$$\underset{\sim}{B}_1 = \underset{\sim}{a}(0)\underset{\sim}{a}^T(0) \quad , \quad \underset{\sim}{B}_2 = \underset{\sim}{a}'(0)\underset{\sim}{a}'(0)$$
$$\underset{\sim}{B}_3 = \underset{\sim}{a}(1)\underset{\sim}{a}^T(1) \quad , \quad \underset{\sim}{B}_4 = \underset{\sim}{a}'(1)\underset{\sim}{a}'^T(1)$$

$$\underset{\sim}{B}_5 = \underset{\sim}{a}(1)\underset{\sim}{a}'^T(1)$$

In terms of the matrices defined in (10), Eq. (9) is written as:

$$\sum_{i=1}^{L} \delta \underset{\sim}{Y}^{*(i)T}\{L^3\underset{\sim}{A}_3 + \frac{\lambda^2}{L}\underset{\sim}{A}_1 - L\underset{\sim}{M}_p\}\underset{\sim}{Y}^{(i)}$$

$$+ \delta \underset{\sim}{Y}^{*(1)T}\{k_1\underset{\sim}{B}_1 + k_2L^2\underset{\sim}{B}_2\}Y^{(1)}$$

$$+ \delta \underset{\sim}{Y}^{*(L)T}\{k_3\underset{\sim}{B}_3 + k_4L^2\underset{\sim}{B}_4 + k_5\underset{\sim}{B}_5\}\underset{\sim}{Y}^{(L)} = 0 \tag{11}$$

where the matrix $\underset{\sim}{M}_p$ is defined as

$$\underset{\sim}{M}_p = \int_0^1 P^{(i)}\underset{\sim}{a}'\underset{\sim}{a}'^T d\xi + \int_0^1 P^{(i)'}\underset{\sim}{a}\underset{\sim}{a}'^T d\xi \ . \tag{12}$$

To proceed further, it is necessary to know the specific form of P(x). As we have mentioned earlier, three different forms of P(x) will be considered.

<u>CASE I. P(x) = P, a Constant.</u> In this case, one has

$$\begin{aligned} P(x) &= P^{(i)}(\xi) = P \\ P'(x) &= LP^{(i)'}(\xi) = 0 \end{aligned} \tag{13}$$

Thus

$$\underset{\sim}{M}_p = P\int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'d\xi = P\underset{\sim}{A}_2 \ . \tag{14}$$

<u>CASE II. P(x) = q(1-x).</u>

$$\begin{aligned} P(x) &= P^{(i)}(\xi) = \frac{q}{L}(L-i+1-\xi) \\ P^{(i)'}(\xi) &= -\frac{q}{L} \end{aligned} \tag{15}$$

Thus,

$$\underset{\sim}{M}_p = \frac{q}{L}\{[L - (i-1)]\int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi - \int_0^1 \xi\underset{\sim}{a}'\underset{\sim}{a}'^T d\xi$$

or

$$\underset{\sim}{M}_p = \frac{q}{L}\{[L - (i-1)]\underset{\sim}{A}_2 - \underset{\sim}{A}_5\} \tag{16}$$

CASE III. $P(x) = q_o/2(1-x)^2$.

$$p^{(i)}(\xi) = \frac{q_o}{2L^2}[(L-i+1)^2 - 2(L-i+1)\xi + \xi^2]$$

$$p^{(i)\prime}(\xi) = -\frac{q_o}{L^2}[(L-i+1) - \xi] \tag{17}$$

Thus,

$$\underset{\sim}{M}_p = \frac{q_o}{2L^2}\{(L-i+1)^2\int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi - 2(L-i+1)\int_0^1 \xi\underset{\sim}{a}'\underset{\sim}{a}'^T d\xi + \int_0^1 \xi^2\underset{\sim}{a}'\underset{\sim}{a}'^T d\xi\}$$

$$- \frac{q_o}{L^2}\{(L-i+1)\int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi - \int_0^1 \xi\underset{\sim}{a}'\underset{\sim}{a}'^T d\xi\}$$

or,

$$\underset{\sim}{M}_p = \frac{q_o}{2L^2}\{(L-i+1)^2\underset{\sim}{A}_2 - 2(L-i+1)\underset{\sim}{A}_5 + \underset{\sim}{A}_7\}$$

$$- \frac{q}{L^2}\{(L-i+1)\underset{\sim}{A}_2 - \underset{\sim}{A}_5\} . \tag{18}$$

With $\underset{\sim}{M}_p$ defined for all three cases in Eqs. (14), (16), and (18) respectively, one can now assemble Eq. (11) into a global matrix equation. Introducing the global generalized coordinate vectors $\underset{\sim}{Y}$ and $\underset{\sim}{Y}^*$ as:

$$\underset{\sim}{Y}^T = \{ Y_1^{(1)} \; Y_2^{(1)} \; Y_3^{(1)} \; Y_4^{(1)} \; Y_3^{(2)} \; Y_4^{(2)} \ldots\ldots Y_3^{(L)} \; Y_4^{(L)} \}$$

$$\underset{\sim}{Y}^{*T} = \{Y_1^{*(1)} \; Y_2^{*(1)} \; Y_3^{*(1)} \; Y_4^{*(1)} \; Y_3^{*(2)} \; Y_4^{*(2)} \ldots\ldots Y_3^{*(L)} \; Y_4^{*(L)}\} \tag{19}$$

Eq. (11) now can be written in terms of $\underset{\sim}{Y}$ and $\delta\underset{\sim}{Y}^*$ as

$$\delta\underset{\sim}{Y}^{*T}\{\underset{\sim}{K} - \lambda^2\underset{\sim}{M}\}\underset{\sim}{Y} = 0 \tag{20}$$

where the global matrices $\underset{\sim}{K}$ and $\underset{\sim}{M}$ are formed by properly placing the local matrices defined in Eqs. (10) according to the correspondence between the local and global generalized coordinates indicated in Eqs. (19). Now since $\delta\underset{\sim}{Y}^*$ are not subject to any constraint conditions, Eq. (18) reduces to

$$(\underset{\sim}{K} - \lambda^2\underset{\sim}{M})\underset{\sim}{Y} = 0 \tag{21}$$

which is solved for the eigenvalue λ and the eigenvector $\underset{\sim}{Y}$.

IV. NUMERICAL RESULTS AND DISCUSSION. It is well known that the eigenvalue λ dictates the stability of the column: a pure imaginary λ is associated with a stable vibration, a real λ with instability of divergence, and a complex λ with instability of flutter [10].

Only cantilevered columns will be considered here. It will be seen that in all three loading cases, the cantilevered columns reaches an instability condition of flutter.

CASE I. $P(x) = P$ = Constant. The characteristic equation in close form was obtained by Beck [3] as

$$2\lambda^2 + Q^2 + 2\lambda^2\cosh\alpha\cos\beta + Q\lambda\sinh\alpha\sin\beta = 0 \tag{22}$$

where

$$\begin{aligned} \alpha^2 &= \sqrt{\lambda^2 + \frac{Q^2}{4}} + \frac{Q}{2} \\ \beta^2 &= \sqrt{\lambda^2 + \frac{Q^2}{4}} + \frac{Q}{2} \end{aligned} \tag{23}$$

For a given Q, the eigenvalue λ can be calculated from Eq. (22) and there are an infinite number of λ solutions for each Q. Eq. (22) is solved for two lowest branches of λ using an iterative procedure. The results are given in Table I. The critical load thus obtained is

$$Q_{CR} = 2.0318\pi^2 = 20.053$$

which agrees well with the value obtained originally by Beck as Q_{CR} = 20.05. The results for four lowest eigenvalues presently obtained using our finite element-unconstrained variational formulations are also shown in Table I. The first two branches obviously agree well with those from the exact characteristic equation. It should be pointed out that the numerical solutions to the Beck problem given in Reference [4] appear to be inaccurate. A plot of the eigenvalue curve showing the coalescense of the two lowest branches is given in Figure 2. The data from Reference [4] are indicated by small circles. The fact that these data points do not fall on a smooth curve further add to the doubt about their accuracy.

CASE II. $P(x) = q(1-x)$. The numerical values of the four lowest eigenvalues up to the first critical load are given in Table II which the critical load is shown to be

$$q_{CR} = 4.0591\pi^2$$

compared with data given by Leipholz as $4.1238\pi^2 = 40.7$ [11] and again as $4.2058\pi^2 = 41.51$ [12]. The coalescence of the first two branches of eigenvalues is again shown in Figure 2.

CASE III. $P(x) = q_o/2(1-x)^2$. Similar data for this case are presented in Table III and in Figure 3. The critical load of flutter is obtained as

$$q_{oCR} = 15.2687\pi^2$$

In comparison, the value obtained by Hauger was $q_{oCR} = 158.2 = 16.092\pi^2$ [6] and that by Leipholz, $q_{oCR} = 150.80 = 15.279\pi^2$ [12].

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TABLE I. NUMERICAL VALUES OF FIRST TWO LOWEST EIGENVALUES OF A CANTILEVERED COLUMN WITH P(x) = P, A CONSTANT

	Q/π^2	0.	0.5	1.0	1.5	2.0	2.0318
λ_1	Present Results	3.5160	4.2072	5.1462	6.5546	9.8256	11.0167
	Exact	3.5160	4.2072	5.1461	6.5546	9.8282	
	Timo. & Gere	3.4894	5.0325	5.4158	6.6939	9.6702	
λ_2	Present Results	22.0356	20.4590	18.6410	16.3684	12.2599	11.0167
	Exact	22.0345	20.4578	18.6395	16.3665	12.2545	
	Timo. & Gere	21.7579	20.2266	17.9290	15.9143	9.9678	
	λ_3	61.7209	59.8566	57.9304	55.9366	53.8689	53.7348
	λ_4	121.0745	119.0413	116.9720	114.8648	112.7177	112.5797

TABLE II. NUMERICAL VALUES OF FIRST FOUR LOWEST EIGENVALUES OF A CANTILEVERED COLUMN WITH $P(x) = q(1-x)$

Q/π^2	0.	1.0	2.0	3.0	4.0	4.05907
λ_1	3.5160	4.2079	5.1499	6.5660	9.8811	11.0315
λ_2	22.0356	20.4587	18.6399	16.3664	12.2289	11.0315
λ_3	61.7209	59.8516	57.9059	55.8772	53.7547	53.6261
λ_4	121.0745	119.0382	116.9586	114.8332	112.6595	112.5295

TABLE III. NUMERICAL VALUES OF FIRST FOUR LOWEST EIGENVALUES OF A CANTILEVERED COLUMN WITH $P(x) = q_0(1-x)^2/2$

Q/π^2	0.	4.0	8.0	12.0	15.26866
λ_1	3.5160	4.3170	5.4413	7.2148	11.4874
λ_2	22.0356	20.4456	18.5880	16.1747	11.4874
λ_3	61.7209	59.5525	57.2512	54.7927	52.6444
λ_4	121.0745	118.6056	116.0544	113.4143	111.1856

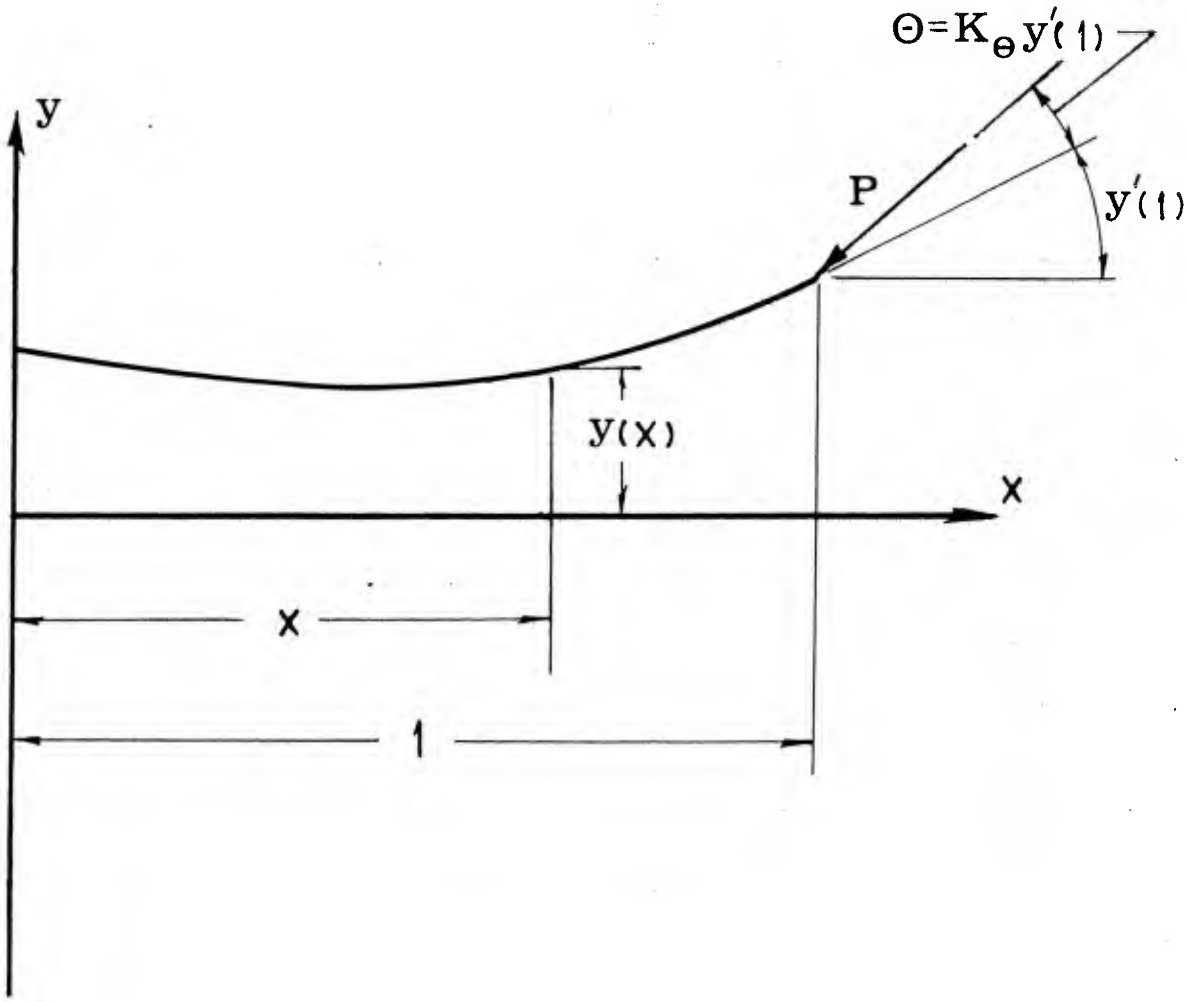

Figure 1. Boundary Condition Associated with a Follower Force: Constant of Tangency K_θ.

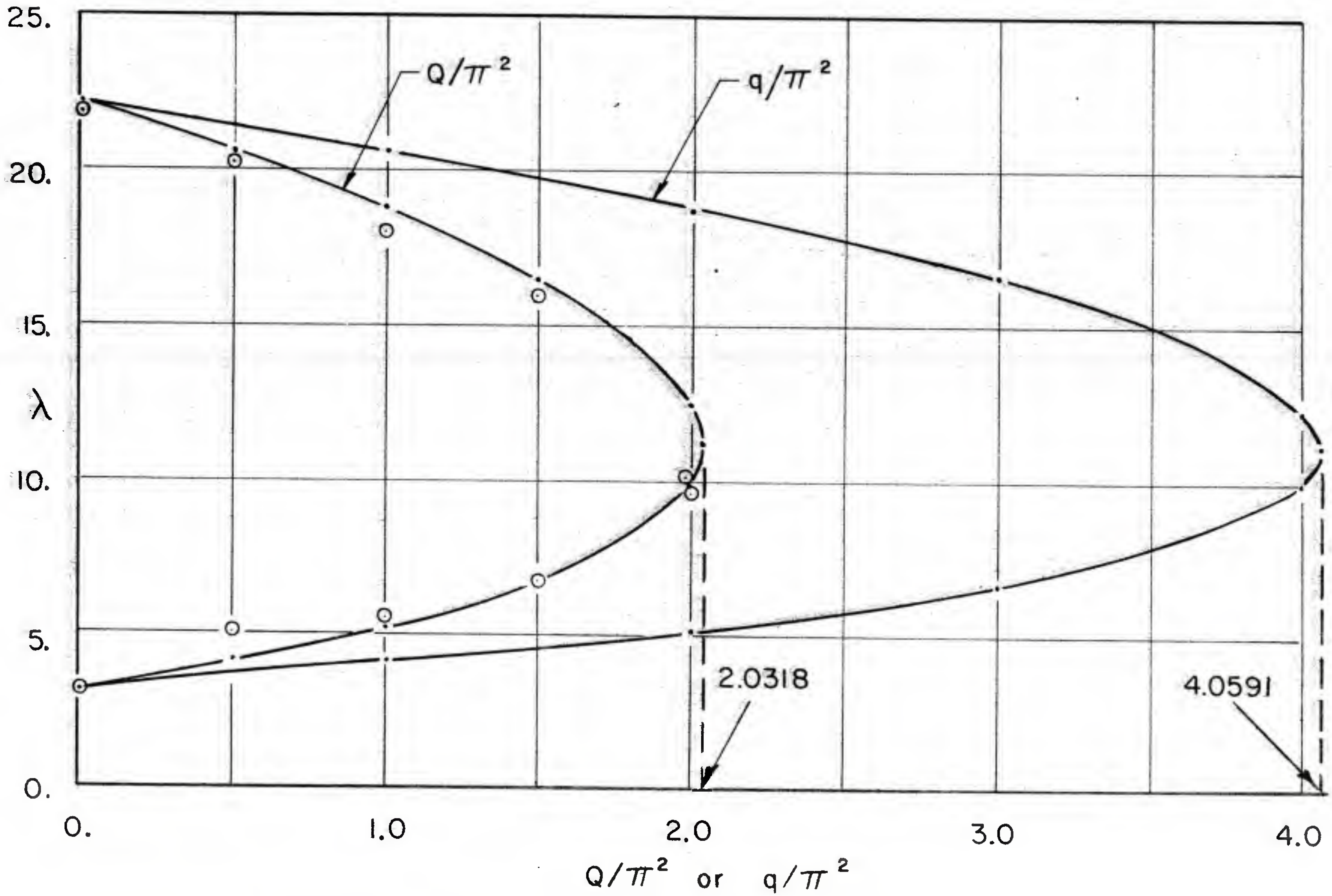

Figure 2. Two Lowest Eigenvalues vs. Load Parameters Q/π^2 and q/π^2.
A Cantilevered Column of Case I and II.
(Data shown in dots (∘) are from Reference [4])

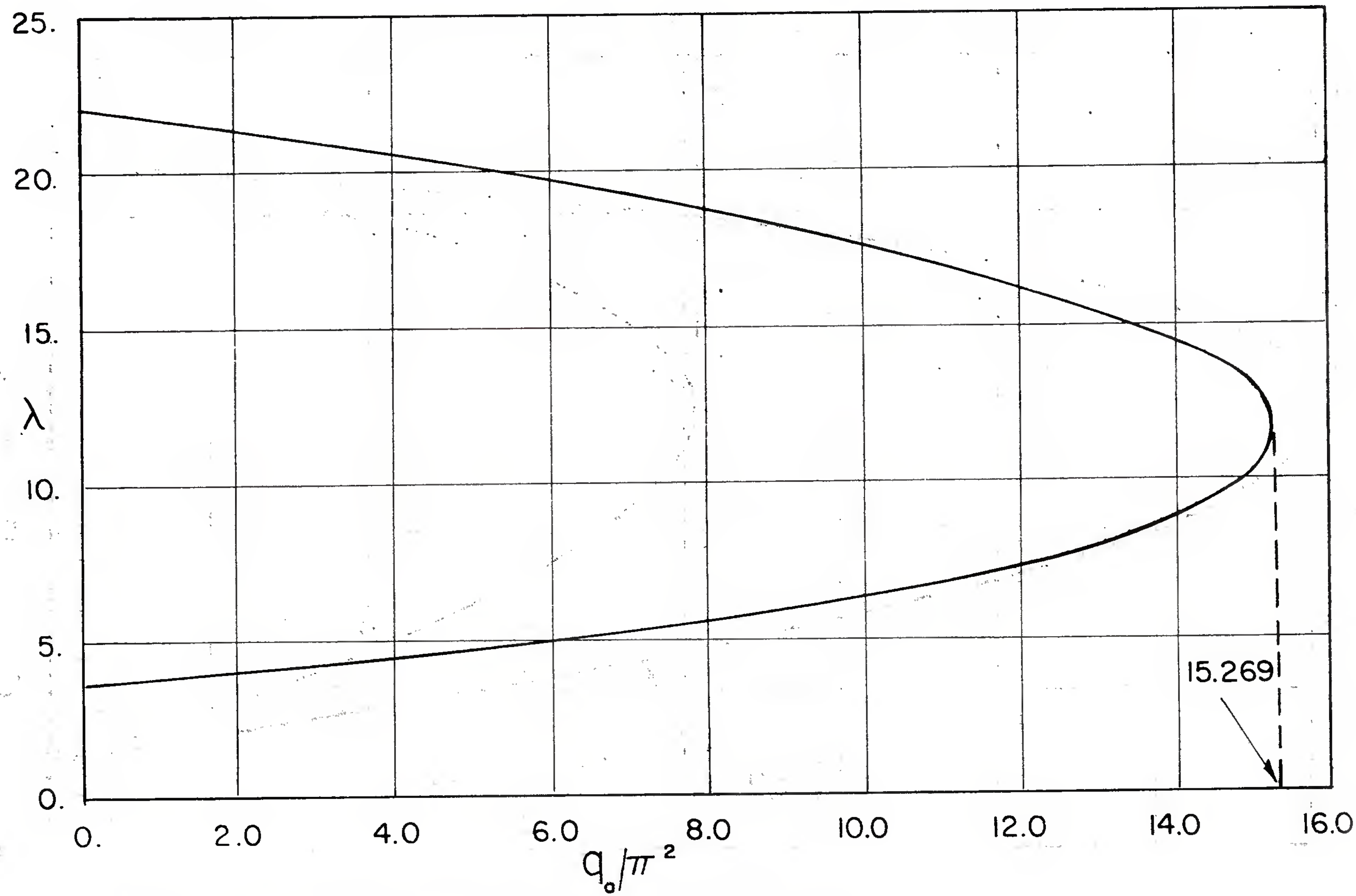

Figure 3. Two Lowest Eigenvalues vs. Load Parameter q_o/π^2
A Cantilevered Column of Case III.

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